Seminar on Stochastic Differential Equations

Summer Semester 2024

Description

Many dynamical processes in the real world are typically modelled as solutions to ordinary differential equations, however there comes a limit as to how accurate these models can be based off of noise in data or random fluctuations in a stochastic dynamical system. The inclusion of a stochastic term in the model may yield a more accurate and life-like model for said system. A trivial (and also fundamental) example of such a system would be the modelling of pollen particles in a solution; the random motions of the pollen particles are due to water molecules hitting the pollen. This process would later be named *Brownian motion*, a fundamental stochastic process that was studied by Einstein [2] and formalised Wiener [9]. Such ordinary differential equations with additional stochastic terms are called *Stochastic Differential Equations* (SDEs for short). Equations of these sorts were initiated by K. Itô [3] and the eponymous Itô formula established a calculus rule for processes driven by the dynamics of SDEs.

The power of SDEs is apparent from its widespread use in financial mathematics as a tool construct arbitrage-free prices for financial options and derivatives and to even model the price of stocks and interest rates. More recently, SDEs have seen an explosion in interest from the machine learning community as the popularity of diffusion models has increased. SDEs have also found fruitful application in many other diverse areas: such as sampling, MCMC, biological processes, interacting particle systems, filtering theory, and even modelling current in electric circuits.

Key information

Time:	Date and time to be announced.
Format:	Room number and location to be announced.
Possible Modules:	B.Mat.3441: Seminar on applied and mathematical stochastics
	M.Mat.4841: Seminar on applied and mathematical stochastics
	B.Mat.3447: Seminar on statistical foundations of data science
	M.Mat.4847: Seminar on statistical foundations of data science
Instructors:	Dr. Alexander Lewis
Intended Audience:	**Advanced Bachelor and beginning Master students**
Language:	English

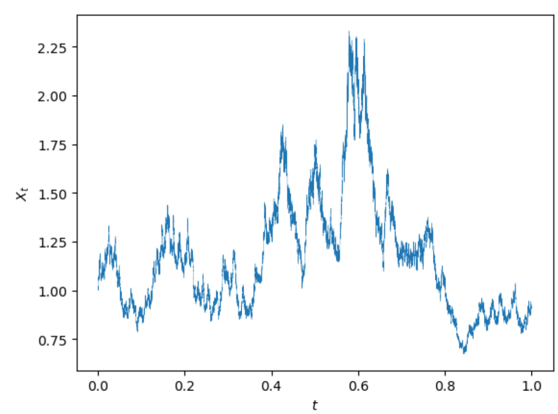
Prerequisites

Participants must have successfully attended

- Measure & probability theory (B.Mat.1400).

It is further considered helpful to have attended either

- Stochastics (B.Mat.2410).



Registration

Please email Alexander Lewis (alexander.lewis@uni-goettingen.de) by the end of the day on **Thursday 21st March 2024** to pre-register for the seminar. Please indicate your interest to give a seminar talk and include information about relevant courses you have taken in your email.

A preliminary meeting, including an introduction to the seminar topic and discussion of the distribution of the presentations will be held virtually on **Friday 22nd March 2024 at 14:15 - 16:00** on Zoom. Meeting room information will be provided to registered students. The seminar has a limited number of participants. In case of outnumbering, participants will be chosen based on the information provided in their preregistration email.

If you cannot attend the meeting or if you have any further questions, please contact me directly with the above email address.

List of Presentations

This is a tentative list of presentations:

1. **Construction of the Itô Integral:** Defining the Riemann–Stieltjes integral and quadratic variation. Using these tools to construct Itô integral of simple processes by taking the left endpoint in L^2. Suggested reading: [4, Section 4.1].
2. **The Itô formula:** Deriving the Itô formula. Providing examples of the application of Itô's formula. Integration by parts and also the multidimensional version of Itô's formula. Suggested reading: [4, Section 4.6].
3. **Stochastic differential equations:** Examples of solving differential equations, such as geometric Brownian motion, Ornstein–Uhlenbeck processes, Brownian bridge. Langevin equations and stationary distributions. Suggested reading: [4, Section 5.1-5.3].
4. **Semigroups and generators of diffusion processes:** Defining diffusion semigroups as an operator. Proving the infinitesimal generator of a general diffusion process. Defining invariant distributions. Suggested reading: [7, Section 7.3] or [8, III.6] (More advanced).
5. **Solving PDEs via diffusions:** Beginning with the Kolmogorov backward equation. Connecting the solution of PDEs to diffusions via the Feynman–Kac formula. Suggested reading: [5, Section 10.9] or [7, Section 8.1-8.2].
6. **Stochastic calculus for jump processes:** Defining jump processes such as the compensated Poisson process. Looking into the modified Itô formula for jump processes and integration. Suggested reading: [5, Section 6.5] or [4, Section 9.3-9.4].
7. **Local time:** Considering reflections of a Brownian motion on a boundary. Defining the local time and deriving the Tanaka formula. Explaining the general connection between reflections and Neumann boundary conditions. Suggested reading: [5, Section 8.6] or [1, Section 7].
8. **Change of measure**: Define the Radon–Nikodym derivative and apply it to a Brownian motion with drift. Prove the Cameron–Martin–Girsanov theorem for general drift. For further investigation, explore the Cameron–Martin space. Suggested reading: [7, Section 8.6] or [4, Section 10.3-10.4].
9. **Financial Markets**: Describe how stochastic processes can be used to emulate the dynamics of a stock market. Explain arbitrage and conditions on the underlying processes to avoid it. Suggested reading: [4, Section 11.1-11.3].
10. **The Black–Scholes model:** Using geometric Brownian motion to describe the dynamics of a stock's price. Solving the Black–Scholes formula to derive a rational price for a European call option. Suggested reading: [6, Section 4.1].

11. **Double Wiener-Itô Integrals**: Describe both Wiener and Itô's ideas for defining a double integral with respect to Brownian motion. Define the correct formulation via off-diagonal step functions. Suggested reading: [5, Section 9.1-9.2].

12. **Approximation of SDEs**: Defining the Euler and Milstein algorithms for the approximation of an SDE. Proving weak and mean-square convergence of the approximations to the solution of the SDE. Suggested reading: [6, Section 3.4].

Other topics may be requested if there is interest.

References

[1] Chung, K. L. and Williams, R. J. (1990). *Introduction to stochastic integration*, volume 2. Springer.

[2] Einstein, A. (1905). Über die von der molekularkinetischen theorie der wärme geforderte bewegung von in ruhenden flüssigkeiten suspendierten teilchen. *Annalen der physik*, 4.

[3] Itô, K. (1944). 109. Stochastic Integral. *Proceedings of the Imperial Academy*, 20(8):519–524.

[4] Klebaner, F. C. (2012). *Introduction to stochastic calculus with applications*. World Scientific Publishing Company.

[5] Kuo, H.-H. (2006). *Introduction to Stochastic Integrals*. Springer.

[6] Mikosch, T. (1998). *Elementary stochastic calculus with finance in view*. World scientific.

[7] Oksendal, B. (2013). *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media.

[8] Rogers, L. C. and Williams, D. (2000). *Diffusions, markov processes, and martingales: Volume 1, foundations*, volume 1. Cambridge university press.

[9] Wiener, N. (1938). The homogeneous chaos. *American Journal of Mathematics*, 60(4):897–936.