



DFG-SNF Research Group FOR916

Statistical Regularization and Qualitative Constraints

Shuheng Zhou

Restricted Eigenvalue Conditions on Subgaussian Random Matrices

Preprint FOR916 09-18

Preprint-Series of the Research Group FOR916

Restricted Eigenvalue Conditions on Subgaussian Random Matrices

Shuheng Zhou

Seminar für Statistik, Department of Mathematics, ETH Zürich, CH-8092, Switzerland

December 20, 2009

Abstract

It is natural to ask: what kinds of matrices satisfy the Restricted Eigenvalue (RE) condition? In this paper, we associate the RE condition (Bickel-Ritov-Tsybakov 09) with the *complexity* of a subset of the sphere in \mathbb{R}^p , where p is the dimensionality of the data, and show that a class of random matrices with independent rows, but not necessarily independent columns, satisfy the RE condition, when the sample size is above a certain lower bound. Here we explicitly introduce an additional covariance structure to the class of random matrices that we have known by now that satisfy the Restricted Isometry Property as defined in Candès and Tao 05 (and hence the RE condition), in order to compose a broader class of random matrices for which the RE condition holds. In this case, tools from geometric functional analysis in characterizing the intrinsic *low-dimensional* structures associated with the RE condition has been crucial in analyzing the sample complexity and understanding its statistical implications for high dimensional data.

Keywords. High dimensional data, Statistical estimation, ℓ_1 minimization, Sparsity, Lasso, Dantzig selector, Restricted Isometry Property, Restricted Eigenvalue conditions, Subgaussian random matrices

1 Introduction

In a typical high dimensional setting, the number of variables p is much larger than the number of observations n . This challenging setting appears in linear regression, signal recovery, covariance selection in graphical modeling, and sparse approximations. In this paper, we consider recovering $\beta \in \mathbb{R}^p$ in the following linear model:

$$Y = X\beta + \epsilon, \tag{1.1}$$

where X is an $n \times p$ design matrix, Y is a vector of noisy observations and ϵ being the noise term. The design matrix is treated as either fixed or random. We assume throughout this paper that $p \geq n$ (i.e. high-dimensional) and $\epsilon \sim N(0, \sigma^2 I_n)$. Throughout this paper, we assume that the columns of X have ℓ_2 norms in the order of \sqrt{n} , which holds with an overwhelming probability when X is a random design that we shall consider.

The restricted eigenvalue (RE) conditions as formalized by [Bickel et al. \(2009\)](#)¹ are among the weakest and hence the most general conditions in literature imposed on the Gram matrix in order to guarantee nice statistical properties for the Lasso and the Dantzig selector; for example, under this condition, they derived bounds on ℓ_2 prediction loss and on ℓ_p , where $1 \leq p \leq 2$, loss for estimating the parameters for both the Lasso and the Dantzig selector in both linear regression and nonparametric regression models. From now on, we refer to their conditions in general as the RE condition. Before we elaborate upon the RE condition, we need some notation and some more definitions to put this condition in perspective.

Consider the linear regression model in (1.1). For a chosen penalization parameter $\lambda_n \geq 0$, regularized estimation with the ℓ_1 -norm penalty, also known as the Lasso ([Tibshirani, 1996](#)) or the Basis Pursuit ([Chen et al., 1998](#)) refers to the following convex optimization problem

$$\widehat{\beta} = \arg \min_{\beta} \frac{1}{2n} \|Y - X\beta\|_2^2 + \lambda_n \|\beta\|_1, \quad (1.2)$$

where the scaling factor $1/(2n)$ is chosen by convenience.

The Dantzig selector ([Candès and Tao, 2007](#)), for a given $\lambda_n \geq 0$, is defined as

$$(DS) \quad \arg \min_{\widehat{\beta} \in \mathbb{R}^p} \|\widehat{\beta}\|_1 \quad \text{subject to} \quad \left\| \frac{1}{n} X^T (Y - X\widehat{\beta}) \right\|_{\infty} \leq \lambda_n. \quad (1.3)$$

For an integer $1 \leq s \leq p/2$, we refer to a vector $\beta \in \mathbb{R}^p$ with at most s non-zero entries as an s -sparse vector. Let $\beta_T \in \mathbb{R}^{|T|}$, be a subvector of $\beta \in \mathbb{R}^p$ confined to T . One of the common properties of the Lasso and the Dantzig selector is: for an appropriately chosen λ_n , for a vector $v := \widehat{\beta} - \beta$, where β is an s -sparse vector and $\widehat{\beta}$ is the solution from either the Lasso or the Dantzig selector, it holds with high probability (cf. Section C)

$$\|v_{I^c}\|_1 \leq k_0 \|v_I\|_1, \quad (1.4)$$

where $I \subset \{1, \dots, p\}$, $|I| \leq s$ is the support of β , $k_0 = 1$ for the Dantzig selector, and for the Lasso it holds for $k_0 = 3$; see [Bickel et al. \(2009\)](#) and [Candès and Tao \(2007\)](#) in case columns of X have ℓ_2 norm \sqrt{n} . We use v_{T_0} to always represent the subvector of $v \in \mathbb{R}^p$ confined to T_0 , which corresponds to the locations of the s largest coefficients of v in absolute values: then (1.4) implies that (see Proposition 1.4)

$$\|v_{T_0^c}\|_1 \leq k_0 \|v_{T_0}\|_1. \quad (1.5)$$

We are now ready to introduce the Restricted Eigenvalue assumption that is formalized in [Bickel et al. \(2009\)](#). In Section 3, we show the convergence rate on ℓ_p for $p = 1, 2$ for both the Lasso and the Dantzig selector under this condition for the purpose of completeness.

Assumption 1.1. (Restricted Eigenvalue assumption $RE(s, k_0, X)$ ([Bickel et al., 2009](#))) For some integer $1 \leq s \leq p$ and a positive number k_0 , the following holds:

$$\frac{1}{K(s, k_0, X)} \triangleq \min_{\substack{J_0 \subseteq \{1, \dots, p\}, \\ |J_0| \leq s}} \min_{\substack{v \neq 0, \\ \|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1}} \frac{\|Xv\|_2}{\sqrt{n} \|v_{J_0}\|_2} > 0. \quad (1.6)$$

¹We note the authors have defined two such conditions, for which we show are equivalent except on the constant defined within each definition; see Proposition A.1 and Proposition A.2 in Section A.3 for details.

Definition 1.1. Throughout this paper, we say that a vector $v \in \mathbb{R}^p$ is admissible to (1.6), or equivalently to (1.12), for a given $k_0 > 0$ as defined therein, if $v \neq 0$ and for some $J_0 \in \{1, \dots, p\}$ such that $|J_0| \leq s$, it holds that $\|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1$. Now it is clear that if v is admissible to (1.6), or equivalently to (1.12), (1.5) holds (cf. Proposition 1.4).

If $RE(s, k_0, X)$ is satisfied with $k_0 \geq 1$, then the square submatrices of size $\leq 2s$ of $X^T X/n$ are necessarily positive definite (see Bickel et al. (2009)). We note the “universality” of this condition as it is not tailored to any particular set J_0 . We also note that given such a universality condition, it is sufficient to check if for all $v \neq 0$ that is admissible to (1.6) and for $K(s, k_0, X) > 0$, the following inequality

$$\frac{\|Xv\|_2}{\sqrt{n}} \geq \frac{\|v_{T_0}\|_2}{K(s, k_0, X)} > 0 \quad (1.7)$$

holds, where T_0 corresponds to locations of the s largest coefficients of v in absolute values, as (1.7) is both necessary and also sufficient to guarantee that (1.6) holds; See Proposition 1.4 for details.

A special class of design matrices that satisfy the RE condition are the random design matrices. This is shown in a large body of work in the high dimensional setting, for example (Candès et al., 2006; Candès and Tao, 2005, 2007; Baraniuk et al., 2008; Mendelson et al., 2008; Adamczak et al., 2009), which shows that a uniform uncertainty principle (UUP, a condition that is stronger than the RE condition, see Bickel et al. (2009)) holds for “generic” or random design matrices for very significant values of s ; roughly speaking, UUP holds when the $2s$ -restricted isometry constant θ_{2s} is small, which we now define. Let X_T , where $T \subset \{1, \dots, p\}$ be the $n \times |T|$ submatrix obtained by extracting columns of X indexed by T .

Definition 1.2. (Candès and Tao, 2005) For each integer $s = 1, 2, \dots$, the s -restricted isometry constant θ_s of X is the smallest quantity such that

$$(1 - \theta_s) \|c\|_2^2 \leq \|X_T c\|_2^2 / n \leq (1 + \theta_s) \|c\|_2^2, \quad (1.8)$$

for all $T \subset \{1, \dots, p\}$ with $|T| \leq s$ and coefficients sequences $(c_j)_{j \in T}$.

It is well known that for a random matrix the UUP holds for $s = O(n/\log(p/n))$ with i.i.d. Gaussian random variables (that is, Gaussian random ensemble, subject to normalizations of columns), the Bernoulli, and in general the subgaussian ensembles (Baraniuk et al., 2008; Mendelson et al., 2008) (cf. Theorem 2.5). Recently, it is shown (Adamczak et al., 2009) that UUP holds for $s = O(n/\log^2(p/n))$ when X is a random matrix composed of columns that are independent isotropic vectors with log-concave densities. Hence this setup only requires $\Theta(\log(p/n))$ or $\Theta(\log^2(p/n))$ observations per nonzero value in β , where Θ hides a very small constant, when n is a nonnegligible fraction of p , in order to perform accurate statistical estimation; we call this level of sparsity as the linear sparsity.

The main purpose of this paper is to extend the family of random matrices from the i.i.d. subgaussian ensemble Ψ (cf. (1.10)), which are now well known to satisfy the UUP condition and hence the RE condition under linear sparsity, to a larger family of random matrices $X := \Psi \Sigma^{1/2}$, where Σ is assumed to behave sufficiently nicely in the sense that it satisfies certain restricted eigenvalue conditions to be defined in Section 1.1. Thus we have explicitly introduced the additional covariance structure Σ to the columns of Ψ in generating X . In Theorem 1.6, we show that X satisfies the RE condition with overwhelming probability

once we have $n \geq Cs \log(cp/s)$, where c is an absolute constant and C depends on the restricted eigenvalues of Σ (cf. (1.19)), when Σ satisfies the restricted eigenvalue assumption to be specified in Section 1.1. We believe such results can be extended to other cases: for example, when X is the composition of a random Fourier ensemble, or randomly sampled rows of orthonormal matrices, see for example Candès and Tao (2006, 2007).

Finally, we show rate of convergence results for the Lasso and the Dantzig selector given such random matrices. Although such results are almost entirely known, we provide a complete analysis for a self-contained presentation. Given these rates of convergence (cf. Theorem 3.1 and Theorem 3.2), one can exploit thresholding algorithms to adjust the bias and get rid of excessive variables selected by an initial estimator relying on ℓ_1 regularized minimization functions, for example, the Lasso or the Dantzig selector; under the UUP or the RE type of conditions, such procedures are shown to select a sparse model, which contains the set of variables in β that are significant in their absolute values; in addition, one can then conduct an ordinary least squares regression on such a sparse model to obtain a final estimator, whose bias is significantly reduced compared to the initial estimators. Such algorithms are proposed and analyzed in a series of papers, for example Candès and Tao (2007); Meinshausen and Yu (2009); Wasserman and Roeder (2009); Zhou (2009).

1.1 Restricted eigenvalue assumption for a random design

We will define the family of random matrices that we consider and the restricted eigenvalue assumption that we impose on such a random design. We need some more definitions.

Definition 1.3. Let Y be a random vector in \mathbb{R}^p ; Y is called isotropic if for every $y \in \mathbb{R}^p$, $\mathbb{E} |\langle Y, y \rangle|^2 = \|y\|_2^2$, and is ψ_2 with a constant α if for every $y \in \mathbb{R}^p$,

$$\| \langle Y, y \rangle \|_{\psi_2} := \inf \{ t : \mathbb{E} \exp(\langle Y, y \rangle^2 / t^2) \leq 2 \} \leq \alpha \|y\|_2. \quad (1.9)$$

The important examples of isotropic, subgaussian vectors are the Gaussian random vector $Y = (h_1, \dots, h_p)$ where $h_i, \forall i$ are independent $N(0, 1)$ random variables, and the random vector $Y = (\varepsilon_1, \dots, \varepsilon_p)$ where $\varepsilon_i, \forall i$ are independent, symmetric ± 1 Bernoulli random variables.

A subgaussian or ψ_2 operator is a random operator $\Gamma : \mathbb{R}^p \rightarrow \mathbb{R}^n$ of the form

$$\Gamma = \sum_{i=1}^n \langle \Psi_i, \cdot \rangle e_i, \quad (1.10)$$

where e_1, \dots, e_n are the canonical basis of \mathbb{R}^n and Ψ_1, \dots, Ψ_n are independent copies of an isotropic ψ_2 vector Ψ_0 on \mathbb{R}^p . Note that throughout this paper, Γ is represented by a random matrix Ψ whose rows are Ψ_1, \dots, Ψ_n . Throughout this paper, we consider a random design matrix X that is generated as follows:

$$X := \Psi \Sigma^{1/2}, \quad \text{where we assume } \Sigma_{jj} = 1, \forall j = 1, \dots, p, \quad (1.11)$$

and Ψ is a random matrix whose rows Ψ_1, \dots, Ψ_n are independent copies of an isotropic ψ_2 vector Ψ_0 on \mathbb{R}^p as in Definition 1.3. For a random design X as in (1.11), we make the following assumption on Σ .

A slightly stronger condition has been originally defined in [Zhou et al. \(2009\)](#) in the context of Gaussian graphical modeling.

Assumption 1.2. Restricted eigenvalue condition $RE(s, k_0, \Sigma)$. Suppose $\Sigma_{jj} = 1, \forall j = 1, \dots, p$, and for some integer $1 \leq s \leq p$ and a positive number k_0 , the following condition holds,

$$\frac{1}{K(s, k_0, \Sigma)} := \min_{\substack{J_0 \subseteq \{1, \dots, p\}, \\ |J_0| \leq s}} \min_{\substack{v \neq 0, \\ \|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1}} \frac{\|\Sigma^{1/2}v\|_2}{\|v_{J_0}\|_2} > 0. \quad (1.12)$$

We note that similar to the case in Assumption 1.1, it is sufficient to check if for $v \neq 0$ that is admissible to (1.12) and for $K(s, k_0, \Sigma) > 0$, that the following inequality

$$\|\Sigma^{1/2}v\|_2 \geq \frac{\|v_{T_0}\|_2}{K(s, k_0, \Sigma)} > 0 \quad (1.13)$$

holds, where T_0 corresponds to locations of the s largest coefficients of v in absolute values. Formally, we have

Proposition 1.4. Let $1 \leq s \leq p/2$ be an integer and $k_0 > 0$. Suppose $\delta \neq 0$ is admissible to (1.12), or equivalently to (1.6), in the sense of Definition 1.1; then

$$\|\delta_{T_0^c}\|_1 \leq k_0 \|\delta_{T_0}\|_1; \quad (1.14)$$

Hence (1.13) is both necessary and sufficient to guarantee that (1.12) holds. Similarly (1.7) is a necessary and sufficient condition for (1.6) to hold. Moreover, suppose that Σ satisfies Assumption 1.2, then for δ that is admissible to (1.12), we have

$$\|\Sigma^{1/2}\delta\|_2 \geq \frac{\|\delta_{J_0}\|_2}{K(s, k_0, \Sigma)} > 0.$$

We now define

$$\sqrt{\rho_{\min}(m)} := \min_{\substack{\|t\|_2=1 \\ |\text{supp}(t)| \leq m}} \|\Sigma^{1/2}t\|_2, \quad (1.15)$$

$$\sqrt{\rho_{\max}(m)} := \max_{\substack{\|t\|_2=1 \\ |\text{supp}(t)| \leq m}} \|\Sigma^{1/2}t\|_2, \quad (1.16)$$

where we assume that $\sqrt{\rho_{\max}(m)}$ is a constant for $m \leq p/2$. If $RE(s, k_0, \Sigma)$ is satisfied with $k_0 \geq 1$, then the square submatrices of size $\leq 2s$ of Σ are necessarily positive definite (see [Bickel et al. \(2009\)](#)); hence throughout this paper, we also assume that

$$\rho_{\min}(2s) > 0. \quad (1.17)$$

Note that when Ψ is a Gaussian random matrix with i.i.d. $N(0, 1)$ random variables, X as in (1.11) corresponds to a random matrix with independent rows, such that each row is a random vector that follows a multivariate normal distribution $N(0, \Sigma)$:

$$X \text{ has i.i.d. rows } \sim N(0, \Sigma), \text{ where we assume } \Sigma_{jj} = 1, \forall j = 1, \dots, p. \quad (1.18)$$

Finally, we need the following notation. For a set $V \subset \mathbb{R}^p$, we let $\text{conv } V$ denote the convex hull of V . For a finite set Y , the cardinality is denoted by $|Y|$. Let B_2^p and S^{p-1} be the unit Euclidean ball and the unit sphere respectively.

1.2 The main theorem

Throughout this section, we assume that Σ satisfies (1.12) and (1.16) for $m = s$. We assume $k_0 > 0$ and it is understood to be the same quantity throughout our discussion. Let us define

$$\bar{C} = 3(2 + k_0)K(s, k_0, \Sigma)\sqrt{\rho_{\max}(s)}, \quad (1.19)$$

where $k_0 > 0$ is understood to be the same as in (1.20). Our main result in Theorem 1.6 roughly says that for a random matrix $X := \Psi\Sigma^{1/2}$, which is the product of a random subgaussian ensemble Ψ and a fixed positive semi-definite matrix $\Sigma^{1/2}$, the RE condition will be satisfied with overwhelming probability, given n that is sufficiently large (cf. (1.21)). Before introducing the theorem formally, we define the class of vectors E_s , for a particular integer $1 \leq s \leq p/2$, that are relevant to the RE Assumption 1.1 and 1.2. For any given subset $J_0 \subset \{1, \dots, p\}$ such that $|J_0| \leq s$, we consider the set of vectors δ such that

$$\|\delta_{J_0^c}\|_1 \leq k_0 \|\delta_{J_0}\|_1 \quad (1.20)$$

holds for some $k_0 > 0$, subject to a normalization condition such that $\Sigma^{1/2}\delta \in S^{p-1}$; we then define the set E'_s as unions of all vectors that satisfy the cone constraint as in (1.20) with respect to any index set $J_0 \subset \{1, \dots, p\}$ such that $|J_0| \leq s$;

$$E'_s = \left\{ \delta : \left\| \Sigma^{1/2}\delta \right\|_2 = 1 \text{ s.t. } \exists J_0 \subseteq \{1, \dots, p\} \text{ s.t. } |J_0| \leq s \text{ and (1.20) holds} \right\}.$$

We now define a even broader set: let δ_{T_0} be the subvector of δ confined to the locations of its s largest coefficients:

$$E_s = \left\{ \delta : \left\| \Sigma^{1/2}\delta \right\|_2 = 1 \text{ s.t. } \|\delta_{T_0^c}\|_1 \leq k_0 \|\delta_{T_0}\|_1 \text{ holds,} \right\}$$

Remark 1.5. *It is clear from Proposition 1.4 that $E'_s \subset E_s$ for the same $k_0 > 0$.*

Theorem 1.6 is the main contribution of this paper.

Theorem 1.6. *Set $1 \leq n \leq p$, $0 < \theta < 1$, and $s \leq p/2$. Let Ψ_0 be an isotropic ψ_2 random vector on \mathbb{R}^p with constant α as in Definition 1.3 and Ψ_1, \dots, Ψ_n be independent copies of Ψ_0 . Let Ψ be a random matrix in $\mathbb{R}^{n \times p}$ whose rows are Ψ_1, \dots, Ψ_n . Let Σ satisfy (1.12) and (1.16). If n satisfies for \bar{C} as defined in (1.19)*

$$n > \frac{c'\alpha^4}{\theta^2} \max(\bar{C}^2 s \log(5ep/s), 9 \log p), \quad (1.21)$$

then with probability at least $1 - 2 \exp(-\bar{c}\theta^2 n/\alpha^4)$, we have for all $\delta \in E_s$,

$$1 - \theta \leq \frac{\|\Psi \Sigma^{1/2}\delta\|_2}{\sqrt{n}} \leq 1 + \theta, \quad \text{and} \quad (1.22)$$

$$\forall \rho_i, \quad 1 - \theta \leq \frac{\|\Psi \rho_i\|_2}{\sqrt{n}} \leq 1 + \theta, \quad (1.23)$$

where ρ_1, \dots, ρ_p are column vectors of $\Sigma^{1/2}$, and $c', \bar{c} > 0$ are absolute constants.

We now state some immediate consequences of Theorem 1.6. Consider the random design $X = \Psi \Sigma^{1/2}$ as defined in Theorem 1.6. It is clear when all columns of X have an Euclidean norm close to \sqrt{n} , as guaranteed by (1.23) for $0 < \theta < 1$ that is small, it makes sense to discuss the RE condition in the form of (1.6). We now define the following event \mathcal{R} on a random design X , which provides an upper bound on $K(s, k_0, X)$ for a given $k_0 > 0$, when X satisfies Assumption $RE(s, k_0, X)$:

$$\mathcal{R}(\theta) := \left\{ X : RE(s, k_0, X) \text{ holds with } 0 < K(s, k_0, X) \leq \frac{K(s, k_0, \Sigma)}{1 - \theta} \right\} \quad (1.24)$$

Under Assumption 1.2, we consider the set of vectors $u := \Sigma^{1/2} \delta$, where $\delta \neq 0$ is admissible to (1.12), and show a uniform bound on the concentration of each individual random variable of the form $\|\Gamma u\|_2^2 := \|X \delta\|_2^2$ around its mean. By Proposition 1.4, we have $\|u\|_2 = \|\Sigma^{1/2} \delta\|_2 > 0$. We can now apply (1.22) to each $(\delta / \|\Sigma^{1/2} \delta\|_2) \neq 0$, which belongs to E'_s and hence E_s (see Remark 1.5), and conclude that

$$0 < (1 - \theta) \left\| \Sigma^{1/2} \delta \right\|_2 \leq \frac{\|X \delta\|_2}{\sqrt{n}} \leq (1 + \theta) \left\| \Sigma^{1/2} \delta \right\|_2 \quad (1.25)$$

hold for all $\delta \neq 0$ that is admissible to (1.12), with probability at least $1 - 2 \exp(-\bar{c} \theta^2 n / \alpha^4)$. Now the lower bound in (1.25) implies that

$$\frac{\|X \delta\|_2}{\sqrt{n}} \geq (1 - \theta) \left\| \Sigma^{1/2} \delta \right\|_2 \geq (1 - \theta) \frac{\|\delta_{T_0}\|_2}{K(s, k_0, \Sigma)} > 0, \quad (1.26)$$

where T_0 is the locations of largest coefficients of t in absolute values. Hence (1.23) and event $\mathcal{R}(\theta)$ hold simultaneously, with probability at least $1 - 2 \exp(-\bar{c} \theta^2 n / \alpha^4)$, given (1.13) and Proposition 1.4, so long as n satisfies (1.21).

Remark 1.7. *It is clear that this result generalizes the notion of restricted isometry property (RIP) introduced in Candès and Tao (2005). In particular, when $\Sigma = I$ and δ is s -sparse, (1.8) holds for X with $\theta_s = \theta$, given (1.25).*

2 Proof Theorem 1.6

In this section, we first state a definition and then two lemmas in Section 2.1, from which we show the proof of Theorem 1.6 in Section 2.2. We shall identify the basis with the canonical basis $\{e_1, e_2, \dots, e_p\}$ of \mathbb{R}^p , where $e_i = \{0, \dots, 0, 1, 0, \dots, 0\}$, and it is to be understood that 1 appears in the i th position and 0 appears elsewhere.

Definition 2.1. *For a subset $V \subset \mathbb{R}^p$, we let*

$$\ell_*(V) = \mathbb{E} \sup_{t \in V} \left| \sum_{i=1}^p g_i t_i \right| \quad (2.1)$$

where $t = (t_i)_{i=1}^p \in \mathbb{R}^p$ and g_1, \dots, g_p are independent $N(0, 1)$ Gaussian random variables.

2.1 The complexity measures

The subset Υ that is relevant to our result is a subset of the sphere S^{p-1} such that the linear function $\Sigma^{1/2} : E_s \rightarrow \mathbb{R}^p$ maps $\delta \in E_s$ onto:

$$\Upsilon := \Sigma^{1/2}(E_s) = \{v \in \mathbb{R}^p : v = \Sigma^{1/2}\delta \text{ for some } \delta \in E_s\}. \quad (2.2)$$

We now show a bound on functional of $\ell_*(\Upsilon)$, for which we crucially exploit the cone property of vectors in E_s , the RE condition on Σ , and the bound of $\rho_{\max}(s)$. Lemma 2.2 is one of the main technical contributions of this paper.

Lemma 2.2. (Complexity of a subset of S^{p-1}) *Let Σ satisfy (1.12) and (1.16). Let h_1, \dots, h_p be independent $N(0, 1)$ random variables. Let $1 \leq s \leq p/2$ be an integer. Then*

$$\ell_*(\Upsilon) := \mathbb{E} \sup_{y \in \Upsilon} \left| \sum_{i=1}^p h_i y_i \right| = \mathbb{E} \sup_{\delta \in E_s} \left| \langle h, \Sigma^{1/2} \delta \rangle \right| \leq \bar{C} \sqrt{s \log(cp/s)} \quad (2.3)$$

where \bar{C} is defined in (1.19) and $c = 5e$.

Remark 2.3. *We will also show in our fundamental proof for the zero-mean Gaussian random ensemble with covariance matrix being Σ , where such complexity measure is used exactly in Section D. There we also give explicit constants.*

Now let $\Sigma^{1/2} := (\rho_{ij})$ and ρ_1, \dots, ρ_p denote its p column vectors. By definition of $\Sigma = (\Sigma^{1/2})^2$, it holds that $\|\rho_i\|_2^2 = \sum_{j=1}^p \rho_{ij}^2 = \Sigma_{ii} = 1$, for all $i = 1, \dots, p$. Thus we have the following.

Lemma 2.4. *Let $\Phi = \{\rho_1, \dots, \rho_p\}$ be the subset of vectors in S^{p-1} that correspond to columns of $\Sigma^{1/2}$. It holds that $\ell_*(\Phi) \leq 3\sqrt{\log p}$.*

2.2 Proof of Theorem 1.6

The key idea to prove Theorem 1.6 is to apply the powerful Theorem 2.5 as shown in Mendelson et al. (2007, 2008)(Corollary 2.7, Theorem 2.1 respectively) to the subset Υ of the sphere S^{p-1} , as defined in (2.2). As explained in Mendelson et al. (2008), in the context of Theorem 2.5, the functional $\ell_*(\Upsilon)$ is the complexity measure of the set Υ , which measures the extent in which probabilistic bounds on the concentration of each individual random variable of the form $\|\Gamma v\|_2^2$ around its mean can be combined to form a bound that holds uniformly for all $v \in \Upsilon$.

Theorem 2.5. (Mendelson et al., 2007, 2008) *Set $1 \leq n \leq p$ and $0 < \theta < 1$. Let Ψ be an isotropic ψ_2 random vector on \mathbb{R}^p with constant α , and Ψ_1, \dots, Ψ_n be independent copies of Ψ . Let Γ be as defined in (1.10) and let $V \subset S^{p-1}$. If n satisfies*

$$n > \frac{c' \alpha^4}{\theta^2} \ell_*(V)^2, \quad (2.4)$$

Then with probability at least $1 - \exp(-\bar{c}\theta^2 n/\alpha^4)$, for all $v \in V$, we have

$$1 - \theta \leq \|\Gamma v\|_2 / \sqrt{n} \leq 1 + \theta, \quad (2.5)$$

where $c', \bar{c} > 0$ are absolute constants.

It is clear that (1.22) follows immediately from Theorem 2.5 by having $V = \Upsilon$, given Lemma 2.2. In fact, we can now finish proving Theorem 1.6 by applying Theorem 2.5 twice, by having $V = \Upsilon$ and $V = \Phi$ respectively: the lower bound on n is obtained by applying the upper bounds on $\ell_*(\Upsilon)$ as given in Lemma 2.2 and on $\ell_*(\Phi)$ as in Lemma 2.4. We then apply the union bound to bound the probability of the bad events when (2.5) does not hold for some $v \in \Upsilon$ or some $v \in \Phi$ respectively. \square

3 ℓ_p convergence for the Lasso and the Dantzig selector

Throughout this section, we assume that $0 < \theta < 1$, and $c', \bar{c} > 0$ are absolute constants. Conditioned on the random design as in (1.11) satisfying properties as guaranteed in Theorem 1.6, we proceed to treat X as a deterministic design, for which both the RE condition as described in (1.24) and condition $\mathcal{F}(\theta)$ defined as below hold,

$$\mathcal{F}(\theta) := \left\{ X : \forall j = 1, \dots, p, 1 - \theta \leq \frac{\|X_j\|_2}{\sqrt{n}} \leq 1 + \theta \right\}, \quad (3.1)$$

where X_1, \dots, X_p are the column vectors of X : Formally, we consider the set $\mathcal{X} \ni X$ of random designs that satisfy both condition $\mathcal{R}(\theta)$ and $\mathcal{F}(\theta)$, for some $0 < \theta < 1$. By Theorem 1.6, we have for n satisfy the lower bound in (1.21),

$$\mathbb{P}(\mathcal{X}) := \mathbb{P}(\mathcal{R}(\theta) \cap \mathcal{F}(\theta)) \geq 1 - 2 \exp(-\bar{c}\theta^2 n / \alpha^4).$$

It is clear that on \mathcal{X} , Assumption 1.2 holds for Σ . We now bound the correlation between the noise and covariates of X for $X \in \mathcal{X}$, where we also define a constant $\lambda_{\sigma, a, p}$ which is used throughout the rest of this paper. For each $a \geq 0$, for $X \in \mathcal{F}(\theta)$, let

$$\mathcal{T}_a := \left\{ \epsilon : \left\| \frac{X^T \epsilon}{n} \right\|_{\infty} \leq (1 + \theta) \lambda_{\sigma, a, p}, \text{ where } X \in \mathcal{F}(\theta), \text{ for } 0 < \theta < 1 \right\}, \quad (3.2)$$

where $\lambda_{\sigma, a, p} = \sigma \sqrt{1 + a} \sqrt{(2 \log p)/n}$, where $a \geq 0$; we have (cf. Proposition C.1)

$$\mathbb{P}(\mathcal{T}_a) \geq 1 - (\sqrt{\pi \log pp^a})^{-1}; \quad (3.3)$$

In fact, for such a bound to hold, we only need $\frac{\|X_j\|_2}{\sqrt{n}} \leq 1 + \theta, \forall j$ to hold in $\mathcal{F}(\theta)$. We note that constants in the theorems are not optimized.

Theorem 3.1. (Estimation for the Lasso) *Set $1 \leq n \leq p$, $0 < \theta < 1$, and $a > 0$. Let $s < p/2$. Consider the linear model in (1.1) with random design $X := \Psi \Sigma^{1/2}$, where $\Psi_{n \times p}$ is a subgaussian random matrix as defined in Theorem 1.6. and Σ satisfies (1.12) and (1.16). Let $\hat{\beta}$ be an optimal solution to the Lasso as in (1.2) with $\lambda_n \geq 2(1 + \theta) \lambda_{\sigma, a, p}$. Suppose that n satisfies for \bar{C} as in (1.19),*

$$n > \frac{c' \alpha^4}{\theta^2} \max(\bar{C}^2 s \log(5ep/s), 9 \log p). \quad (3.4)$$

Then with probability at least $\mathbb{P}(\mathcal{X} \cap \mathcal{T}_a) \geq 1 - 2 \exp(-\bar{c}\theta^2 n / \alpha^4) - \mathbb{P}(\mathcal{T}_a^c)$, we have for $B \leq 4K^2(s, 3, \Sigma)/(1 - \theta)^2$ and $k_0 = 3$,

$$\left\| \hat{\beta} - \beta \right\|_2 \leq 2B \lambda_n \sqrt{s}, \quad \text{and} \quad \left\| \hat{\beta} - \beta \right\|_1 \leq B \lambda_n s. \quad (3.5)$$

Theorem 3.2. (Estimation for the Dantzig selector) Set $1 \leq n \leq p$, $0 < \theta < 1$, and $a > 0$. Let $s < p/2$. Consider the linear model in (1.1) with random design $X := \Psi \Sigma^{1/2}$, where $\Psi_{n \times p}$ is a subgaussian random matrix as defined in Theorem 1.6. and Σ satisfies (1.12) and (1.16). Let $\hat{\beta}$ be an optimal solution to the Dantzig selector as in (1.3) where $\lambda_n \geq (1 + \theta)\lambda_{\sigma,a,p}$. Suppose that n satisfies for \bar{C} as in (1.19),

$$n > \frac{c' \alpha^4}{\theta^2} \max(\bar{C}^2 s \log(5ep/s), 9 \log p). \quad (3.6)$$

then with probability at least $\mathbb{P}(\mathcal{X} \cap \mathcal{T}_a) \geq 1 - 2 \exp(-\bar{c}\theta^2 n/\alpha^4) - \mathbb{P}(\mathcal{T}_a^c)$, we have for $B \leq 4K^2(s, 1, \Sigma)/(1 - \theta)^2$ and $k_0 = 1$,

$$\left\| \hat{\beta} - \beta \right\|_2 \leq 3B\lambda_n \sqrt{s}, \text{ and } \left\| \hat{\beta} - \beta \right\|_1 \leq 2B\lambda_n s. \quad (3.7)$$

Proofs are given in Section C.

Acknowledgments. Research is supported by the Swiss National Science Foundation (SNF) Grant 20PA21-120050/1. The author is extremely grateful to Guillaume Lecué for his careful reading of the manuscript and for his many constructive and insightful comments that have lead to a significant improvement of the presentation of this paper. The author would also like to thank Olivier Guédon, Alain Pajor, and Larry Wasserman for helpful conversations, and Roman Vershynin for providing a reference.

A Some preliminary propositions

In this section, we first prove Proposition 1.4, in Section A.1, which is used throughout the rest of the paper. We then present a simple decomposition for vectors $\delta \in E_s$ and show some immediate implications, which we shall need in the proofs for Lemma 2.2, Theorem 3.1 and Theorem 3.2.

A.1 Proof of Proposition 1.4

For each δ that is admissible to (1.12), there exists a subset of indices $J_0 \subseteq \{1, \dots, p\}$ such that both $|J_0| \leq s$ and $\|\delta_{J_0^c}\|_1 \leq k_0 \|\delta_{J_0}\|_1$ hold. This immediately implies that (1.14) holds for $k_0 > 0$,

$$\|\delta_{T_0^c}\|_1 = \|\delta\|_1 - \|\delta_{T_0}\|_1 \leq \|\delta\|_1 - \|\delta_{J_0}\|_1 = \|\delta_{J_0^c}\|_1 \leq k_0 \|\delta_{J_0}\|_1 \leq k_0 \|\delta_{T_0}\|_1$$

due to the maximality of $\|\delta_{T_0}\|_1$ among all $\|\delta_{J_0}\|_1$ for $J_0 \subseteq \{1, \dots, p\}$ such that $|J_0| \leq s$. This immediately implies that $E'_s \subset E_s$.

We now show that (1.13) is a necessary and sufficient condition for (1.12) to hold; the same argument applies to the RE conditions on X . Suppose (1.13) hold for $\delta \neq 0$; we have for all $J_0 \in \{1, \dots, p\}$ such that $|J_0| \leq s$ and $\|\delta_{J_0^c}\|_1 \leq k_0 \|\delta_{J_0}\|_1$,

$$\left\| \Sigma^{1/2} \delta \right\|_2 \geq \frac{\|\delta_{T_0}\|_2}{K(s, k_0, \Sigma)} \geq \frac{\|\delta_{J_0}\|_2}{K(s, k_0, \Sigma)} > 0, \quad (\text{A.1})$$

where the last inequality is due to the fact that $\|\delta_{J_0}\|_2 > 0$; Suppose $\|\delta_{J_0}\|_2 = 0$ otherwise; then $\|\delta_{J_0^c}\|_1 \leq k_0 \|\delta_{J_0}\|_1 = 0$ would imply that $\delta = 0$, which is a contradiction. Conversely, suppose that (1.12) hold; then (1.13) must also hold, given that T_0 satisfies (1.14) with $|T_0| = s$, and $\delta_{T_0} \neq 0$.

Finally, the ‘‘moreover’’ part holds given Assumption 1.2, in view of (A.1). \square

A.2 Decomposing a vector in E_s

For each $\delta \in E_s$, we decompose δ into a set of vectors $\delta_{T_0}, \delta_{T_1}, \delta_{T_2}, \dots, \delta_{T_K}$ such that T_0 corresponds to locations of the s largest coefficients of δ in absolute values, T_1 corresponds to locations of the s largest coefficients of $\delta_{T_0^c}$ in absolute values, T_2 corresponds to locations of the next s largest coefficients of $\delta_{T_0^c}$ in absolute values, and so on. Hence we have $T_0^c = \bigcup_{k=1}^K T_k$, where $K \geq 1, |T_k| = s, \forall k = 1, \dots, K-1$, and $|T_K| \leq s$. Now for each $j \geq 1$, we have

$$\|\delta_{T_j}\|_2 \leq \sqrt{s} \|\delta_{T_j}\|_\infty \leq \frac{1}{\sqrt{s}} \|\delta_{T_{j-1}}\|_1,$$

where vector $\|\cdot\|_\infty$ represents the largest entry in absolute value in the vector, and hence

$$\begin{aligned} \sum_{k \geq 1} \|\delta_{T_k}\|_2 &\leq s^{-1/2} (\|\delta_{T_0}\|_1 + \|\delta_{T_1}\|_1 + \|\delta_{T_2}\|_1 + \dots) \\ &\leq s^{-1/2} (\|\delta_{T_0}\|_1 + \|\delta_{T_0^c}\|_1) = s^{-1/2} \|\delta\|_1 \end{aligned} \quad (\text{A.2})$$

$$\leq s^{-1/2} (k_0 + 1) \|\delta_{T_0}\|_1 \leq (k_0 + 1) \|\delta_{T_0}\|_2, \quad (\text{A.3})$$

where for (A.3), we have used the fact that for all $\delta \in E_s$

$$\|\delta_{T_0^c}\|_1 \leq k_0 \|\delta_{T_0}\|_1 \quad (\text{A.4})$$

holds. Indeed, for δ such that (A.4) holds, we have by (A.2) and (A.3)

$$\|\delta\|_2 \leq \|\delta_{T_0}\|_2 + \sum_{j \geq 1} \|\delta_{T_j}\|_2 \leq \|\delta_{T_0}\|_2 + s^{-1/2} \|\delta\|_1 \quad (\text{A.5})$$

$$\leq (k_0 + 2) \|\delta_{T_0}\|_2. \quad (\text{A.6})$$

A.3 On the equivalence of two RE conditions

To introduce the second RE assumption by [Bickel et al. \(2009\)](#), we need some more notation. For an integer s such that $1 \leq s \leq p/2$, a vector $v \in \mathbb{R}^p$ and a set of indices $J_0 \subseteq \{1, \dots, p\}$ with $|J_0| \leq s$, denoted by J_1 the subset of $\{1, \dots, p\}$ corresponding to the s largest in absolute value coordinates of v outside of J_0 and defined $J_{01} \triangleq J_0 \cup J_1$.

Assumption A.1. Restricted eigenvalue assumption $RE(s, s, k_0, X)$ ([Bickel et al., 2009](#)). Consider a fixed design. For some integer $1 \leq s \leq p/2$, and a positive number k_0 , the following condition holds:

$$\frac{1}{K(s, s, k_0, X)} := \min_{\substack{J_0 \subseteq \{1, \dots, p\}, \\ |J_0| \leq s}} \min_{\substack{v \neq 0, \\ \|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1}} \frac{\|Xv\|_2}{\sqrt{n} \|v_{J_{01}}\|_2} > 0. \quad (\text{A.7})$$

Assumption A.2. Restricted eigenvalue assumption $RE(s, s, k_0, \Sigma)$ For some integer $1 \leq s \leq p/2$, and a positive number k_0 , the following condition holds:

$$\frac{1}{K(s, s, k_0, \Sigma)} := \min_{\substack{J_0 \subseteq \{1, \dots, p\}, \\ |J_0| \leq s}} \min_{\substack{v \neq 0, \\ \|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1}} \frac{\|\Sigma^{1/2}v\|_2}{\|v_{J_0}\|_2} > 0. \quad (\text{A.8})$$

Proposition A.1. For some integer $1 \leq s \leq p/2$, and for the same $k_0 > 0$, the two sets of RE conditions are equivalent up to a constant $\sqrt{2}$ factor of each other:

$$\frac{K(s, s, k_0, \Sigma)}{\sqrt{2}} \leq K(s, k_0, \Sigma) \leq K(s, s, k_0, \Sigma);$$

Similarly, we have

$$\frac{K(s, s, k_0, X)}{\sqrt{2}} \leq K(s, k_0, X) \leq K(s, s, k_0, X).$$

Proof. It is obvious that for the same $k_0 > 0$, (A.8) implies that the condition as in Definition 1.2 holds with

$$K(s, k_0, \Sigma) \leq K(s, s, k_0, \Sigma).$$

Now, for the other direction, suppose that $RE(s, k_0, \Sigma)$ holds for $K(s, k_0, \Sigma) > 0$. Then for all $v \neq 0$ that is admissible to (1.12), we have by Proposition 1.4,

$$\|\Sigma^{1/2}v\|_2 \geq \frac{\|v_{T_0}\|_2}{K(s, k_0, \Sigma)} > 0, \quad (\text{A.9})$$

where T_0 corresponds to locations of the s largest coefficients of v in absolute values; Now for any $J_0 \subseteq \{1, \dots, p\}$ such that $|J_0| \leq s$, and $\|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1$ holds, we have by (1.12),

$$\|\Sigma^{1/2}v\|_2 \geq \frac{\|v_{J_0}\|_2}{K(s, k_0, \Sigma)} > 0. \quad (\text{A.10})$$

Now it is clear that $J_1 \subset T_0 \cup T_1$, and we have for all $v \neq 0$ that is admissible to (1.12),

$$0 < \|v_{J_0}\|_2^2 = \|v_{J_0}\|_2^2 + \|v_{J_1}\|_2^2 \quad (\text{A.11})$$

$$\leq \|v_{J_0}\|_2^2 + \|v_{T_0}\|_2^2 \quad (\text{A.12})$$

$$\leq 2K^2(s, k_0, \Sigma) \|\Sigma^{1/2}v\|_2^2, \quad (\text{A.13})$$

which immediately implies that for all $v \neq 0$ that is admissible to (1.12),

$$\frac{\|\Sigma^{1/2}v\|_2}{\|v_{J_0}\|_2} \geq \frac{1}{\sqrt{2}K(s, k_0, \Sigma)} > 0.$$

Thus we have that $RE(s, s, k_0, \Sigma)$ condition holds with $K(s, s, k_0, \Sigma) \leq \sqrt{2}K(s, k_0, \Sigma)$. The other set of inequalities follow exactly the same line of arguments. \square

We now introduce the last assumption, for which we need some more notation. For integers s, m such that $1 \leq s \leq p/2$ and $m \geq s, s + m \leq p$, a vector $\delta \in \mathbb{R}^p$ and a set of indices $J_0 \subseteq \{1, \dots, p\}$ with $|J_0| \leq s$, denoted by J_m the subset of $\{1, \dots, p\}$ corresponding to the m largest in absolute value coordinates of δ outside of J_0 and defined $J_{0m} \triangleq J_0 \cup J_m$.

Assumption A.3. Restricted eigenvalue assumption $RE(s, m, k_0, X)$ (Bickel et al., 2009). For some integer $1 \leq s \leq p/2, m \geq s, s + m \leq p$, and a positive number k_0 , the following condition holds:

$$\frac{1}{K(s, m, k_0, X)} := \min_{\substack{J_0 \subseteq \{1, \dots, p\}, \\ |J_0| \leq s}} \min_{\substack{v \neq 0, \\ \|v_{J_0^c}\|_1 \leq k_0 \|v_{J_0}\|_1}} \frac{\|Xv\|_2}{\sqrt{n} \|v_{J_{0m}}\|_2} > 0. \quad (\text{A.14})$$

Proposition A.2. For some integer $1 \leq s \leq p/2, m \geq s, s + m \leq p$, and some positive number k_0 , we have

$$\frac{K(s, m, k_0, X)}{\sqrt{2 + k_0^2}} \leq K(s, k_0, X) \leq K(s, m, k_0, X).$$

Proof. It is clear that $K(s, k_0, X) \leq K(s, m, k_0, X)$ for $m \geq s$. Now suppose that $RE(s, k_0, X)$ holds, we continue from (A.10). We divide J_m into J_1, J_2, \dots , such that J_1 corresponds to locations of the s largest coefficients of $v_{J_0^c}$ in absolute values, J_2 corresponds to locations of the next s largest coefficients of $v_{J_0^c}$ in absolute values, and so on. We first bound $\|v_{J_0^c}\|_2^2$, following essentially the same argument as in Candès and Tao (2007): observe that the k th largest value of $v_{J_0^c}$ obeys

$$|v_{J_0^c}|_{(k)} \leq \|v_{J_0^c}\|_1 / k;$$

Thus we have for δ that is admissible to (A.14),

$$\begin{aligned} \|v_{J_0^c}\|_2^2 &\leq \|v_{J_0^c}\|_1^2 \sum_{j \geq s+1} 1/k^2 \leq s^{-1} \|v_{J_0^c}\|_1^2 \\ &\leq s^{-1} k_0^2 \|v_{J_0}\|_1^2 \leq k_0^2 \|v_{J_0}\|_2^2. \end{aligned}$$

It is clear that $\|J_1\|_2 \leq \|T_0\|_2$, and

$$\begin{aligned} 0 < \|v_{J_{01}}\|_2^2 &\leq \|v_{J_{0m}}\|_2^2 \leq \|v_{J_0}\|_2^2 + \|v_{J_1}\|_2^2 + \|v_{J_{01}^c}\|_2^2 \\ &\leq \|v_{J_0}\|_2^2 + \|v_{J_1}\|_2^2 + k_0^2 \|v_{J_0}\|_2^2 \\ &\leq (1 + k_0^2) \|v_{J_0}\|_2^2 + \|v_{T_0}\|_2^2 \\ &\leq (2 + k_0^2) K^2(s, k_0, X) \|Xv\|_2^2, \end{aligned}$$

which immediately implies that for all $v \neq 0$ that is admissible to (A.14),

$$\frac{\|Xv\|_2}{\|v_{J_{0m}}\|_2} \geq \frac{1}{\sqrt{2 + k_0^2} K(s, k_0, X)} > 0.$$

Thus we have that $RE(s, m, k_0, X)$ condition holds with $K(s, m, k_0, X) \leq \sqrt{2 + k_0^2} K(s, k_0, X)$. \square

B Results on the complexity measures

In this section, in preparation for proving Lemma 2.2 and Lemma 2.4, we first state some well-known definitions and some preliminary results on certain complexity measures on a set V (See Mendelson et al. (2008) for example); we also provide a new result in Lemma B.6.

Definition B.1. Given a subset $U \subset \mathbb{R}^p$ and a number $\varepsilon > 0$, an ε -net Π of U with respect to the Euclidean metric is a subset of points of U such that ε -balls centered at Π covers U :

$$U \subset \bigcup_{x \in \Pi} (x + \varepsilon B_2^p),$$

where $A + B := \{a + b : a \in A, b \in B\}$ is the Minkowski sum of the sets A and B . The covering number $\mathcal{N}(U, \varepsilon)$ is the smallest cardinality of an ε -net of U .

Now it is well-known that there exists an absolute constant $c_1 > 0$ such that for every finite subset $\Pi \subset B_2^p$,

$$\ell_*(\text{conv } \Pi) = \ell_*(\Pi) \leq c_1 \sqrt{\log |\Pi|}. \quad (\text{B.1})$$

The main goal of the rest of this section is to provide a bound on a variation of the complexity measure $\ell_*(V)$, which we will denote with $\tilde{\ell}_*(V)$ throughout this paper, by essentially exploiting a bound similar to (B.1) (cf. Lemma B.6).

Given a set $V \subset \mathbb{R}^p$, we need to also measure $\ell_*(W)$, where W is the subspace of \mathbb{R}^p such that the linear function $\Sigma^{1/2} : V \rightarrow \mathbb{R}^p$ carries $t \in V$ onto:

$$W := \Sigma^{1/2}(V) = \{w \in \mathbb{R}^p : w = \Sigma^{1/2}t \text{ for some } t \in V\}.$$

We denote this new measure with $\tilde{\ell}_*(V)$. Formally,

Definition B.2. For a subset $V \subset \mathbb{R}^p$, we define

$$\tilde{\ell}_*(V) := \ell_*(\Sigma^{1/2}(V)) := \mathbb{E} \sup_{t \in V} \left| \langle t, \Sigma^{1/2}h \rangle \right| := \mathbb{E} \sup_{t \in V} \left| \sum_{i=1}^p g_i t_i \right| \quad (\text{B.2})$$

where $t = (t_i)_{i=1}^p \in \mathbb{R}^p$, and $h = (h_i)_{i=1}^p \in \mathbb{R}^p$ is a random vector with independent $N(0, 1)$ random variables while $g = \Sigma^{1/2}h$ is a random vector with dependent Gaussian random variables.

We prove a bound on this measure in Lemma B.6 after we present some existing results. The subsets that we would like to apply (2.1) and (B.2) are the sets consisting of sparse vectors: let S^{p-1} be the unit sphere in \mathbb{R}^p , for $1 \leq m \leq p$

$$U_m := \{x \in S^{p-1} : |\text{supp}(x)| \leq m\} \quad (\text{B.3})$$

We shall also consider the analogous subset of the Euclidean ball,

$$\tilde{U}_m := \{x \in B_2^p : |\text{supp}(x)| \leq m\} \quad (\text{B.4})$$

The sets U_m and \tilde{U}_m are unions of the unit spheres, and unit balls, respectively, supported on m -dimensional coordinate subspaces of \mathbb{R}^p . The following three lemmas are well-known and mostly standard; See Mendelson et al. (2008) and Ledoux and Talagrand (1991) for example.

Lemma B.3. (Mendelson et al. (2008, Lemma 2.2)) Given $m \geq 1$ and $\varepsilon > 0$. There exists an ε cover $\Pi \subset B_2^m$ of B_2^m with respect to the Euclidean metric such that $B_2^m \subset (1 - \varepsilon)^{-1} \text{conv } \Pi$ and $|\Pi| \leq (1 + 2/\varepsilon)^m$. Similarly, there exists an ε cover of the sphere S^{m-1} , $\Pi' \subset S^{m-1}$ such that $|\Pi'| \leq (1 + 2/\varepsilon)^m$.

Lemma B.4. (Mendelson et al. (2008, Lemma 2.3)) For every $0 < \varepsilon \leq 1/2$ and every $1 \leq m \leq p$, there is a set $\Pi \subset B_2^p$ which is an ε cover of \tilde{U}_m , such that

$$\tilde{U}_m \subset 2 \text{conv } \Pi, \quad \text{where } |\Pi| \leq \left(\frac{5}{2\varepsilon}\right)^m \binom{p}{m} \quad (\text{B.5})$$

Moreover, there exists an ε cover $\Pi' \subset S^{p-1}$ of U_m with cardinality at most $(\frac{5}{2\varepsilon})^m \binom{p}{m}$.

Proof. Consider all subsets $T \subset \{1, \dots, p\}$ with $|T| = m$, it is clear that the required sets in Π and Π' in Lemma B.4 can be obtained by unions of corresponding sets supported on the coordinates from T . By Lemma B.3, the cardinalities of these sets are at most $(5/2\varepsilon)^m \binom{p}{m}$. \square

Lemma B.5. (Ledoux and Talagrand, 1991) Let $X = (X_1, \dots, X_N)$ be Gaussian in \mathbb{R}^p . Then

$$\mathbb{E} \max_{i=1, \dots, N} |X_i| \leq 3\sqrt{\log N} \max_{i=1, \dots, N} \sqrt{\mathbb{E} X_i^2}.$$

We now prove the key lemma that we need for Lemma D.2. The main point of the proof follows the idea from Mendelson et al. (2008): if $U_m \subset 2 \text{conv } \Pi_m$ for $\Pi_m \subset B_2^p$ and there is a reasonable control of the cardinality of Π_m and $\rho_{\max}(m)$ on Σ , then $\tilde{\ell}_*(V)$ is bounded from above.

Lemma B.6. Let Π_m be a $1/2$ -cover of \tilde{U}_m provided by Lemma B.4. Then for $1 \leq m < p/2$ and $c = 5e$, it holds that for $V = U_m$

$$\tilde{\ell}_*(U_m) \leq \tilde{\ell}_*(2 \text{conv } \Pi_m) = 2\tilde{\ell}_*(\Pi_m) \quad \text{where} \quad (\text{B.6})$$

$$\tilde{\ell}_*(\Pi_m) \leq 3\sqrt{m \log c(p/m)} \sqrt{\rho_{\max}(m)}. \quad (\text{B.7})$$

Proof. The first inequality follows from the definition of $\tilde{\ell}_*$ and the fact that

$$V = U_m \subset \tilde{U}_m \subset 2 \text{conv } \Pi_m.$$

The second equality in (B.6) holds due to convexity which guarantees that

$$\begin{aligned} \sup_{y \in \text{conv } \Pi_m} \left| \langle y, \Sigma^{1/2} h \rangle \right| &= \sup_{y \in \Pi_m} \left| \langle y, \Sigma^{1/2} h \rangle \right| \quad \text{and hence} \\ \tilde{\ell}_*(2 \text{conv } \Pi_m) &= 2\tilde{\ell}_*(\text{conv } \Pi_m) = 2\tilde{\ell}_*(\Pi_m). \end{aligned}$$

Thus we have for $c = 5e$

$$\begin{aligned} \tilde{\ell}_*(\text{conv } \Pi_m) = \tilde{\ell}_*(\Pi_m) &:= \mathbb{E} \sup_{t \in \Pi_m} \left| \langle t, \Sigma^{1/2} h \rangle \right| \\ &\leq 3\sqrt{\log |\Pi_m|} \sup_{t \in \Pi_m} \sqrt{\mathbb{E} \left| \langle t, \Sigma^{1/2} h \rangle \right|^2} \\ &\leq 3\sqrt{m \log(5ep/m)} \sup_{t \in \Pi_m} \left\| \Sigma^{1/2} t \right\|_2 \\ &\leq 3\sqrt{m \log c(p/m)} \sqrt{\rho_{\max}(m)} \end{aligned}$$

where we have used Lemma B.5, (B.5) and the bound $\binom{p}{m} \leq \left(\frac{ep}{m}\right)^m$, which is valid for $m < p/2$, and the fact that $\mathbb{E} \left| \langle t, \Sigma^{1/2} h \rangle \right|^2 = \mathbb{E} \left| \langle h, \Sigma^{1/2} t \rangle \right|^2 = \|\Sigma^{1/2} t\|_2^2$. \square

B.1 Proof of Lemma 2.2

It is clear that for all $y \in \mathcal{Y}$, $y = \Sigma^{1/2} \delta$ for some $\delta \in E_s$, hence all equalities in (2.3) hold. We hence focus on bounding the last term. For each $\delta \in E_s$, we decompose δ into a set of vectors $\delta_{T_0}, \delta_{T_1}, \delta_{T_2}, \dots, \delta_{T_K}$ as in Section A.2.

By Proposition 1.4, we have $\|\delta_{T_0}\|_1 \leq k_0 \|\delta_{T_0}\|_2$.

For each index set $T \subset \{1, \dots, p\}$, we let δ_T represent its 0-extended version δ' in \mathbb{R}^p , such that $\delta'_{T^c} = 0$ and $\delta'_T = \delta_T$. For $\delta_T = 0$, it is understood that $\frac{\delta_T}{\|\delta_T\|_2} := 0$ below. Thus we have for all $\delta \in E_s$ and all $h \in \mathbb{R}^p$,

$$\begin{aligned}
\left| \langle h, \Sigma^{1/2} \delta \rangle \right| &= \left| \langle h, \Sigma^{1/2} \delta_{T_0} + \sum_{k \geq 1} \langle h, \Sigma^{1/2} \delta_{T_k} \rangle \rangle \right| \\
&\leq \left| \langle h, \Sigma^{1/2} \delta_{T_0} \rangle \right| + \sum_{k \geq 1} \left| \langle h, \Sigma^{1/2} \delta_{T_k} \rangle \right| \\
&\leq \left| \langle \delta_{T_0}, \Sigma^{1/2} h \rangle \right| + \sum_{k \geq 1} \|\delta_{T_k}\|_2 \left| \left\langle \frac{\delta_{T_k}}{\|\delta_{T_k}\|_2}, \Sigma^{1/2} h \right\rangle \right| \\
&\leq \|\delta_{T_0}\|_2 \left| \left\langle \frac{\delta_{T_0}}{\|\delta_{T_0}\|_2}, \Sigma^{1/2} h \right\rangle \right| + \sum_{k \geq 1} \|\delta_{T_k}\|_2 \sup_{t \in U_s} \left| \langle t, \Sigma^{1/2} h \rangle \right| \\
&\leq \left(\|\delta_{T_0}\|_2 + \sum_{k \geq 1} \|\delta_{T_k}\|_2 \right) \sup_{t \in U_s} \left| \langle t, \Sigma^{1/2} h \rangle \right| \\
&\leq (k_0 + 2)K(s, k_0, \Sigma) \sup_{t \in U_s} \left| \langle h, \Sigma^{1/2} t \rangle \right|, \tag{B.8}
\end{aligned}$$

where we have used the following bounds in (B.9) and (B.10): By Assumption 1.2 and by construction of its corresponding sets T_0, T_1, \dots , we have for all $\delta \in E_s$,

$$\|\delta_{T_0}\|_2 \leq K(s, k_0, \Sigma) \left\| \Sigma^{1/2} \delta \right\|_2 = K(s, k_0, \Sigma) \tag{B.9}$$

$$\sum_{k \geq 1} \|\delta_{T_k}\|_2 \leq (k_0 + 1) \|\delta_{T_0}\|_2 \leq (k_0 + 1)K(s, k_0, \Sigma), \tag{B.10}$$

where we used the bound in (A.3). Thus we have by (B.8) and Lemma B.6

$$\begin{aligned}
\mathbb{E} \sup_{\delta \in E_s} \left| \langle h, \Sigma^{1/2} \delta \rangle \right| &\leq (2 + k_0)K(s, k_0, \Sigma) \mathbb{E} \sup_{t \in U_s} \left| \langle h, \Sigma^{1/2} t \rangle \right| \\
&\leq (2 + k_0)K(s, k_0, \Sigma) \tilde{\ell}_*(U_s) \\
&\leq 3(2 + k_0)K(s, k_0, \Sigma) \sqrt{s \log(cp/s)} \sqrt{\rho_{\max}(s)} \\
&:= \bar{C} \sqrt{s \log(cp/s)}
\end{aligned}$$

by Lemma B.6, where \bar{C} is as defined in (1.19) and $c = 5e$. This proves Lemma 2.2. \square

B.2 Proof of Lemma 2.4

Let h_1, \dots, h_p be independent $N(0, 1)$ Gaussian random variables. We have by Lemma (B.5),

$$\begin{aligned} \ell_*(\Phi) &:= \mathbb{E} \max_{i=1, \dots, p} \left| \sum_{j=1}^p \rho_{ij} h_j \right| \leq 3\sqrt{\log p} \max_{i=1, \dots, p} \sqrt{\mathbb{E} \left(\sum_{j=1}^p \rho_{ij} h_j \right)^2} \\ &= 3\sqrt{\log p} \max_{i=1, \dots, p} \sqrt{\sum_{j=1}^p (\rho_{ij})^2 \mathbb{E} h_j^2} \\ &= 3\sqrt{\log p} \max_{i=1, \dots, p} \sqrt{\Sigma_{ii}} = 3\sqrt{\log p}, \end{aligned}$$

where we used the fact that $\Sigma_{ii} = 1$ for all i and $\sigma(h_j) = 1, \forall j$. \square

C Proofs for Theorems in Section 3

Throughout this section, let $1 > \theta > 0$. Proving both Theorem 3.1 and Theorem 3.2 involves first showing that the optimal solutions to both the Lasso and the Dantzig selector satisfy the cone constraint as in (1.4) for $I = \text{supp } \beta$, for some $k_0 > 0$. Indeed, it holds that $k_0 = 1$ for the Dantzig selector when $\lambda_n \geq (1 + \theta)\lambda_{\sigma, a, p}$, and $k_0 = 3$ for the Lasso when $\lambda_n \geq 2(1 + \theta)\lambda_{\sigma, a, p}$ (cf. Lemma C.2 and (C.14)). These have been shown before, for example, in Bickel et al. (2009) and in Candès and Tao (2007). We included proofs for (Lemma C.2 and (C.14)) for completeness. We then state two propositions for the Lasso estimator and the Dantzig selector respectively under \mathcal{T}_a , where $a > 0$ and $1 > \theta > 0$. We first bound the probability on \mathcal{T}_a^c .

C.1 Bounding \mathcal{T}_a^c

Lemma C.1. *For fixed design X with $\max_j \|X_j\|_2 \leq (1 + \theta)\sqrt{n}$, where $0 < \theta < 1$, we have for \mathcal{T}_a as defined in (3.2), where $a > 0$, $\mathbb{P}(\mathcal{T}_a^c) \leq (\sqrt{\pi \log pp^a})^{-1}$.*

Proof. Define random variables: $Y_j = \frac{1}{n} \sum_{i=1}^n \epsilon_i X_{i,j}$. Note that $\max_{1 \leq j \leq p} |Y_j| = \|X^T \epsilon / n\|_\infty$. We have $\mathbb{E}(Y_j) = 0$ and $\text{Var}((Y_j)) = \|X_j\|_2^2 \sigma^2 / n^2 \leq (1 + \theta)\sigma^2 / n$. Let $c_0 = 1 + \theta$. Obviously, Y_j has its tail probability dominated by that of $Z \sim N(0, \frac{c_0^2 \sigma^2}{n})$:

$$\mathbb{P}(|Y_j| \geq t) \leq \mathbb{P}(|Z| \geq t) \leq \frac{2c_0\sigma}{\sqrt{2\pi nt}} \exp\left(\frac{-nt^2}{2c_0^2\sigma^2}\right).$$

We can now apply the union bound to obtain:

$$\begin{aligned} \mathbb{P} \left(\max_{1 \leq j \leq p} |Y_j| \geq t \right) &\leq p \frac{c_0 \sigma}{\sqrt{nt}} \exp \left(\frac{-nt^2}{2c_0^2 \sigma^2} \right) \\ &= \exp \left(- \left(\frac{nt^2}{2c_0^2 \sigma^2} + \log \frac{t\sqrt{\pi n}}{\sqrt{2}c_0 \sigma} - \log p \right) \right). \end{aligned}$$

By choosing $t = c_0 \sigma \sqrt{1+a} \sqrt{2 \log p/n}$, the right-hand side is bounded by $(\sqrt{\pi \log pp^a})^{-1}$ for $a \geq 0$. \square

C.2 Proof of Theorem 3.1

Let $\widehat{\beta}$ be an optimal solution to the Lasso as in (1.2). $S := \text{supp } \beta$. and

$$v = \widehat{\beta} - \beta.$$

We first show Lemma C.2; we then apply condition $RE(s, k_0, X)$ on v with $k_0 = 3$ under \mathcal{T}_a to show Proposition C.3. Theorem 3.1 follows immediately from Proposition (C.3).

Lemma C.2. *Bickel et al. (2009)* Under condition \mathcal{T}_a as defined in (3.2), $\|v_{S^c}\|_1 \leq 3\|v_S\|_1$ for $\lambda_n \geq 2(1+\theta)\lambda_{\sigma,a,p}$ for the Lasso.

Proof. By the optimality of $\widehat{\beta}$, we have

$$\begin{aligned} \lambda_n \|\beta\|_1 - \lambda_n \|\widehat{\beta}\|_1 &\geq \frac{1}{2n} \|Y - X\widehat{\beta}\|_2^2 - \frac{1}{2n} \|Y - X\beta\|_2^2 \\ &\geq \frac{1}{2n} \|Xv\|_2^2 - \frac{v^T X^T \epsilon}{n} \end{aligned}$$

Hence under condition \mathcal{T}_a as in (3.2), we have for $\lambda_n \geq 2(1+\theta)\lambda_{\sigma,a,p}$,

$$\begin{aligned} \|Xv\|_2^2/n &\leq 2\lambda_n \|\beta\|_1 - 2\lambda_n \|\widehat{\beta}\|_1 + 2 \left\| \frac{X^T \epsilon}{n} \right\|_\infty \|v\|_1 \\ &\leq \lambda_n \left(2\|\beta\|_1 - 2\|\widehat{\beta}\|_1 + \|v\|_1 \right), \end{aligned} \tag{C.1}$$

where by the triangle inequality, and $\beta_{S^c} = 0$, we have

$$\begin{aligned} 0 &\leq 2\|\beta\|_1 - 2\|\widehat{\beta}\|_1 + \|v\|_1 \\ &= 2\|\beta_S\|_1 - 2\|\widehat{\beta}_S\|_1 - 2\|v_{S^c}\|_1 + \|v_S\|_1 + \|v_{S^c}\|_1 \\ &\leq 3\|v_S\|_1 - \|v_{S^c}\|_1. \end{aligned} \tag{C.2}$$

Thus Lemma C.2 holds. \square

We now show Proposition C.3, where except for the ℓ_2 -convergence rate as in (C.5), all bounds have essentially been shown in Bickel et al. (2009) (as Theorem 7.2) under Assumption $RE(s, 3, X)$; The bound on $\|v\|_2$, which as far as the author is aware of, is new; however, this result is indeed also implied by Theorem 7.2 in Bickel et al. (2009) given Proposition A.1 as derived in this paper. We note that the same remark holds for Proposition C.5; see Bickel et al. (2009, Theorem 7.1).

Proposition C.3. (ℓ_p -loss for the Lasso) Suppose that $RE(s, 3, X)$ holds. Let $Y = X\beta + \epsilon$, for ϵ being i.i.d. $N(0, \sigma^2)$ and $\|X_j\|_2 \leq (1 + \theta)\sqrt{n}$. Let $\hat{\beta}$ be an optimal solution to (1.2) with $\lambda_n \geq 2(1 + \theta)\lambda_{\sigma, a, p}$, where $a \geq 0$. Let $v = \hat{\beta} - \beta$. Then on condition \mathcal{T}_a as in (3.2), the following hold for $B_0 = 4K^2(s, 3, X)$

$$\|v_S\|_2 \leq B_0\lambda_n\sqrt{s}, \quad (\text{C.3})$$

$$\|v\|_1 \leq B_0\lambda_n s, \text{ where } \|v_{S^c}\|_1 \leq 3\|v_S\|_1 \quad (\text{C.4})$$

$$\text{and } \|v\|_2 \leq 2B_0\lambda_n\sqrt{s}. \quad (\text{C.5})$$

Proof. The first part of this proof follows that of [Bickel et al. \(2009\)](#). Now under condition \mathcal{T}_a , by (C.1) and (C.2),

$$\begin{aligned} \|Xv\|_2^2/n + \lambda_n\|v\|_1 &\leq \lambda_n(3\|v_S\|_1 - \|v_{S^c}\|_1 + \|v_S\|_1 + \|v_{S^c}\|_1) \\ &= 4\lambda_n\|v_S\|_1 \leq 4\lambda_n\sqrt{s}\|v_S\|_2 \end{aligned} \quad (\text{C.6})$$

$$\leq 4\lambda_n\sqrt{s}K(s, 3, X)\|Xv\|_2/\sqrt{n} \quad (\text{C.7})$$

$$\leq 4K^2(s, 3, X)\lambda_n^2 s + \|Xv\|_2^2/n. \quad (\text{C.8})$$

where (C.7) holds by definition of $RE(s, 3, X)$; Thus we have by (C.8) that

$$\|v_S\|_1 \leq \|v\|_1 \leq 4K^2(s, 3, X)\lambda_n s, \quad (\text{C.9})$$

which implies that (C.4) holds with $B_0 = 4K^2(s, 3, X)$. Now by $RE(s, 3, X)$ and (C.6), we have

$$\|v_S\|_2^2 \leq K^2(s, 3, X)\|Xv\|_2^2/n \leq K^2(s, 3, X)4\lambda_n\sqrt{s}\|v_S\|_2 \quad (\text{C.10})$$

which immediately implies that (C.3) holds.

Finally, we have by (A.5), (C.9), (1.14) and the $RE(s, 3, X)$ condition,

$$\begin{aligned} \|v\|_2 &\leq \|v_{T_0}\|_2 + s^{-1/2}\|v\|_1 \\ &\leq K(s, 3, X)\|Xv\|_2/\sqrt{n} + 4K^2(s, 3, X)\lambda_n\sqrt{s}, \end{aligned} \quad (\text{C.11})$$

$$\leq K(s, 3, X)\sqrt{4\lambda_n\|v_S\|_1} + 4K^2(s, 3, X)\lambda_n\sqrt{s}, \quad (\text{C.12})$$

$$\leq 8\lambda_n K^2(s, 3, X)\sqrt{s}. \quad (\text{C.13})$$

where in (C.11), we crucially exploit the universality of the RE condition; in (C.12), we use the bound in (C.6); and in (C.13), we use (C.9). \square

C.3 Proof of Theorem 3.2

Let $\hat{\beta}$ be an optimal solution to the Dantzig selector as into (1.3). Let $S := \text{supp } \beta$. and

$$v = \hat{\beta} - \beta.$$

We first show Lemma C.4; we then apply condition $RE(s, k_0, X)$ to v with $k_0 = 1$ under \mathcal{T}_a to show Proposition C.5. Theorem 3.2 follows from immediately from Proposition (C.5).

Lemma C.4. (*Candès and Tao (2007)*) Under condition \mathcal{T}_a , $\|v_{S^c}\|_1 \leq \|v_S\|_1$ for $\lambda_n \geq (1+\theta)\lambda_{\sigma,a,p}$, where $a \geq 0$ and $0 < \theta < 1$ for the Dantzig selector.

Proof. Clearly the true vector β is feasible to (1.3), as

$$\left\| \frac{1}{n} X^T (Y - X\beta) \right\|_\infty = \left\| \frac{1}{n} X^T \epsilon \right\|_\infty \leq (1+\theta)\lambda_{\sigma,a,p} \leq \lambda_n,$$

hence by the optimality of $\hat{\beta}$,

$$\|\hat{\beta}\|_1 \leq \|\beta\|_1.$$

Hence it holds under for $v = \hat{\beta} - \beta$ that

$$\|\beta\|_1 - \|v_S\|_1 + \|v_{S^c}\|_1 \leq \|\beta + v\| = \|\hat{\beta}\|_1 \leq \|\beta\|_1 \quad (\text{C.14})$$

and hence v obeys the cone constraint as desired. \square

Proposition C.5. (*ℓ_p -loss for the Dantzig selector*) Suppose that $RE(s, 1, X)$ holds. Let $Y = X\beta + \epsilon$, for ϵ being i.i.d. $N(0, \sigma^2)$ and $\|X_j\|_2 \leq (1+\theta)\sqrt{n}$. Let $\hat{\beta}$ be an optimal solution to (1.3) with $\lambda_n \geq (1+\theta)\lambda_{\sigma,a,p}$, where $a \geq 0$ and $0 < \theta < 1$. Then on condition \mathcal{T}_a as in (3.2), the following hold with $B_1 = 4K^2(s, 1, X)$

$$\|v_S\|_2 \leq B_1 \lambda_n \sqrt{s}, \quad (\text{C.15})$$

$$\|v\|_1 \leq 2B_1 \lambda_n s, \quad \text{where } \|v_{S^c}\|_1 \leq \|v_S\|_1 \quad (\text{C.16})$$

$$\text{and } \|v\|_2 \leq 3B_1 \lambda_n \sqrt{s}. \quad (\text{C.17})$$

Remark C.6. See comments in front of Proposition C.3.

Proof of Proposition C.5. Our proof follows that of Bickel et al. (2009). Let $\hat{\beta}$ as an optimal solution to (1.3). Let $v = \hat{\beta} - \beta$ and let \mathcal{T}_a hold for $a > 0$ and $0 < \theta < 1$. By the constraint of (1.3), we have

$$\left\| \frac{1}{n} X^T X v \right\|_\infty \leq \left\| \frac{1}{n} X^T (Y - X\hat{\beta}) \right\|_\infty + \left\| \frac{1}{n} X^T \epsilon \right\|_\infty \leq 2\lambda_n.$$

and hence by Lemma C.4, we have

$$\begin{aligned} \|Xv\|_2^2 / n &= \frac{v^T X^T X v}{n} \leq \left\| \frac{1}{n} v^T X^T X \right\|_\infty \|v\|_1 \leq 2\lambda_n \|v\|_1 \\ &\leq 4\lambda_n \|v_S\|_1 \leq 4\lambda_n \sqrt{s} \|v_S\|_2. \end{aligned} \quad (\text{C.18})$$

We now apply condition $RE(s, k_0, X)$ on v with $k_0 = 1$ to obtain

$$\|v_S\|_2^2 \leq K^2(s, 1, X) \|Xv\|_2^2 / n \leq K^2(s, 1, X) 4\lambda_n \sqrt{s} \|v_S\|_2, \quad (\text{C.19})$$

which immediately implies that (C.15) holds. Hence (C.16) holds with $B_1 = 4K^2(s, 1, X)$ given (C.19) and

$$\|v_{S^c}\|_1 \leq \|v_S\|_1 \leq 4K^2(s, 1, X) \lambda_n s. \quad (\text{C.20})$$

Finally, we have by (A.5), (C.20), (1.14) and the $RE(s, 3, X)$ condition,

$$\begin{aligned} \|v\|_2 &\leq \|v_{T_0}\|_2 + s^{-1/2} \|v\|_1 \\ &\leq K(s, 1, X) \|Xv\|_2 / \sqrt{n} + 8K^2(s, 1, X) \lambda_n \sqrt{s}, \end{aligned} \quad (\text{C.21})$$

$$\leq K(s, 1, X) \sqrt{4\lambda_n \|v_S\|_1} + 8K^2(s, 1, X) \lambda_n \sqrt{s}, \quad (\text{C.22})$$

$$\leq 12\lambda_n K^2(s, 1, X) \sqrt{s}. \quad (\text{C.23})$$

where in (C.21), we crucially exploit the universality of the RE condition, and in (C.22), we use the bound in (C.20) and (C.18); and in (C.23), we use (C.20) again. \square

D A fundamental proof for the Gaussian random design

In this section, we state a theorem for the Gaussian random design, following a more fundamental proof given by Raskutti et al. (2009) (cf. Proposition 1). We apply their method and provide a tighter bound on the sample size that is required in order for X to satisfy the RE condition, where X is composed of independent rows with multivariate Gaussian vectors drawn from $N(0, \Sigma)$ as in 1.18. We note that both upper and lower bounds in Theorem D.1 are obtained in a way that is quite similar to how the largest and smallest singular values of a Gaussian random matrix are upper and lower bounded respectively; see for example Davidson and Szarek (2001). The improvement over results in Raskutti et al. (2009) comes from the tighter bound on $\ell_*(\Upsilon)$ as developed in Lemma 2.2. Formally, we have the following.

Theorem D.1. *Set $1 \leq n \leq p$ and $0 < \theta < 1$. Consider a random design X as in(1.11), where Σ satisfies (1.12) and (1.16). Suppose $s < p/2$ and for \bar{C} as in (1.19),*

$$n > \frac{1}{\theta^2} \left(\bar{C} \sqrt{s \log(5ep/s)} + \sqrt{2d \log p} \right)^2 \quad (\text{D.1})$$

for $d > 0$. Then we have with probability at least $1 - 4/p^d$,

$$(1 - \theta - o(1)) \left\| \Sigma^{1/2} \delta \right\|_2 \leq \|X\delta\|_2 / \sqrt{n} \leq (1 + \theta) \left\| \Sigma^{1/2} \delta \right\|_2 \quad (\text{D.2})$$

holds for all $\delta \neq 0$ that is admissible to (1.12), that is, \exists some $J_0 \in \{1, \dots, p\}$ such that $|J_0| \leq s$ and $\|\delta_{J_0^c}\|_1 \leq k_0 \|\delta_{J_0}\|_1$, where $k_0 > 0$.

Proof. We only provide a sketch here; see Raskutti et al. (2009) for details. Using the Slepian's Lemma and its extension by Gordon (1985), the following inequalities have been derived by Raskutti et al. (2009) (cf. Proof of Proposition 1 therein),

$$\begin{aligned} \mathbb{E} \inf_{\delta \in E_s} \|X\delta\|_2 &\geq \mathbb{E} \|g\|_2 - \mathbb{E} \sup_{\delta \in E_s} \left| \langle h, \Sigma^{1/2} \delta \rangle \right|, \\ \mathbb{E} \sup_{\delta \in E_s} \|X\delta\|_2 &\leq \sqrt{n} + \mathbb{E} \sup_{\delta \in E_s} \left| \langle h, \Sigma^{1/2} \delta \rangle \right|, \end{aligned}$$

where g and h are random vectors with i.i.d Gaussian $N(0, 1)$ elements in \mathbb{R}^n and \mathbb{R}^p respectively. Now Lemma D.2 follows immediately, after we plug in the bound as in Lemma 2.2 on

$$\ell_*(\Upsilon) := \mathbb{E} \sup_{\delta \in E_s} \left| \langle h, \Sigma^{1/2} \delta \rangle \right|.$$

Lemma D.2. *Suppose Σ satisfies Assumption 1.2. Then for \bar{C} as in Theorem D.1, we have*

$$\mathbb{E} \inf_{\delta \in E_s} \|X\delta\|_2 \geq \sqrt{n} - o(\sqrt{n}) - \bar{C} \sqrt{s \log(5ep/s)} \quad (\text{D.3})$$

$$\mathbb{E} \sup_{\delta \in E_s} \|X\delta\|_2 \leq \sqrt{n} + \bar{C} \sqrt{s \log(5ep/s)}. \quad (\text{D.4})$$

We then apply the concentration of measure inequality for $\inf_{\delta \in E_s} \|X\delta\|_2$, for which it is well known that the 1-Lipschitz condition holds for $\inf_{\delta \in E_s} \|X\delta\|_2 = \inf_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2$, where A is a matrix with i.i.d. standard normal random variables in $\mathbb{R}^{n \times p}$. Recall a function $f : X \rightarrow Y$ is called 1-Lipschitz condition if for all $x, y \in X$,

$$d_Y(f(x), f(y)) \leq d_X(x, y).$$

Proposition D.3. *View Gaussian random matrix A as a canonical Gaussian vector in \mathbb{R}^{np} . Let $f(A) := \inf_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2$ and $f'(A) := \sup_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2$ be two functions of A from \mathbb{R}^{np} to \mathbb{R} . Then $f, f' : \mathbb{R}^{np} \rightarrow \mathbb{R}$ are 1-Lipschitz:*

$$\begin{aligned} |f(A) - f(B)| &\leq \|A - B\|_2 \leq \|A - B\|_F, \\ |f'(A) - f'(B)| &\leq \|A - B\|_2 \leq \|A - B\|_F. \end{aligned}$$

Finally we apply the concentration of measure in Gauss Space to obtain for $t > 0$,

$$\mathbb{P}(|f(A) - \mathbb{E}f(A)| > t) \leq 2 \exp(-t^2/2), \quad \text{and} \quad (\text{D.5})$$

$$\mathbb{P}(|f'(A) - \mathbb{E}f'(A)| > t) \leq 2 \exp(-t^2/2). \quad (\text{D.6})$$

Now it is clear that with probability at least $1 - 4/p^d$, where $d > 0$, we have for $X = A\Sigma^{1/2}$

$$\begin{aligned} \inf_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2 &=: f(A) \geq \mathbb{E} \inf_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2 - \sqrt{2d \log p} \\ &\geq \sqrt{n} - o(\sqrt{n}) - \bar{C} \sqrt{s \log(5ep/s)} - \sqrt{2d \log p}, \end{aligned}$$

which we denote as event \mathcal{F} , and

$$\begin{aligned} \sup_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2 &=: f'(A) \leq \mathbb{E} \sup_{\delta \in E_s} \|A\Sigma^{1/2}\delta\|_2 + \sqrt{2d \log p} \\ &\leq \sqrt{n} + \bar{C} \sqrt{s \log(5ep/s)} + \sqrt{2d \log p}, \end{aligned}$$

which we denote as \mathcal{F}' . Now it is clear that (D.2) holds on $\mathcal{F} \cap \mathcal{F}'$, given (D.1). \square

References

- ADAMCZAK, R., LITVAK, A. E., PAJOR, A. and TOMCZAK-JAEGERMANN, N. (2009). Restricted isometry property of matrices with independent columns and neighborly polytopes by random sampling. arXiv:0904.4723v1.
- BARANIUK, R. G., DAVENPORT, M., DEVORE, R. A. and WAKIN, M. B. (2008). A simple proof of the restricted isometry property for random matrices. *Constructive Approximation* **28** 253–263.
- BICKEL, P. J., RITOV, Y. and TSYBAKOV, A. B. (2009). Simultaneous analysis of Lasso and Dantzig selector. *The Annals of Statistics* **37** 1705–1732.
- CANDÈS, E., ROMBERG, J. and TAO, T. (2006). Stable signal recovery from incomplete and inaccurate measurements. *Communications in Pure and Applied Mathematics* **59** 1207–1223.
- CANDÈS, E. and TAO, T. (2005). Decoding by Linear Programming. *IEEE Trans. Info. Theory* **51** 4203–4215.
- CANDÈS, E. and TAO, T. (2006). Near optimal signal recovery from random projections: Universal encoding strategies? *IEEE Trans. Info. Theory* **52** 5406–5425.
- CANDÈS, E. and TAO, T. (2007). The Dantzig selector: statistical estimation when p is much larger than n . *Annals of Statistics* **35** 2313–2351.
- CHEN, S. S., DONOHO, D. L. and SAUNDERS, M. A. (1998). Atomic decomposition by basis pursuit. *SIAM Journal on Scientific and Statistical Computing* **20** 33–61.
- DAVIDSON, K. R. and SZAREK, S. (2001). Local operator theory, random matrices and banach spaces. *Handbook on the Geometry of Banach spaces* **1** 317–366.
- GORDON, Y. (1985). Some inequalities for gaussian processes and applications. *Israel Journal of Mathematics* **50** 265–289.
- LEDOUX, M. and TALAGRAND, M. (1991). *Probability in Banach Spaces: Isoperimetry and processes*. Springer.
- MEINSHAUSEN, N. and YU, B. (2009). Lasso-type recovery of sparse representations for high-dimensional data. *Annals of Statistics* **37** 246–270.
- MENDELSON, S., PAJOR, A. and TOMCZAK-JAEGERMANN, N. (2007). Reconstruction and subgaussian operators in asymptotic geometric analysis. *Geometric and Functional Analysis* **17** 1248–1282.
- MENDELSON, S., PAJOR, A. and TOMCZAK-JAEGERMANN, N. (2008). Uniform uncertainty principle for bernoulli and subgaussian ensembles. *Constructive Approximation* **28** 277–289.
- RASKUTTI, G., WAINWRIGHT, M. and YU, B. (2009). Minimax rates of estimation for high-dimensional linear regression over ℓ_q -balls. In *Allerton Conference on Control, Communication and Computer*. Longer version in arXiv:0910.2042v1.pdf.

- TIBSHIRANI, R. (1996). Regression shrinkage and selection via the Lasso. *J. Roy. Statist. Soc. Ser. B* **58** 267–288.
- WASSERMAN, L. and ROEDER, K. (2009). High dimensional variable selection. *The Annals of Statistics* **37** 2178–2201.
- ZHOU, S. (2009). Thresholding procedures for high dimensional variable selection and statistical estimation. In *Advances in Neural Information Processing Systems 22*. MIT Press.
- ZHOU, S., VAN DE GEER, S. and BÜHLMANN, P. (2009). Adaptive Lasso for high dimensional regression and gaussian graphical modeling. ArXiv:0903.2515.